having an even number of circles (with $2k$ points on both the A and the B circles, and i running from 1 to $2k-1$) : the points A_{2k}, A_1, B_1, B_{2k} must also lie on a circle.

For the second claim we will see that the sides of the hexagon $C_1C_2C_3C_4C_5C_6$ are tangent to a conic whose foci are the centres, call them A and B, of the circles A_i and B_i ; by Brianchon's theorem, the lines joining opposite vertices of the hexagon, namely C_1C_4, C_2C_5, and C_3C_6, must be concurrent. To this end, we wish to show that for each i, the unique conic with foci A and B that is tangent to the line C_iC_{i-1} (joining the centres of consecutive circles) coincides with the unique conic with those foci that is tangent to the line C_iC_{i+1}. Note that the conic will be an ellipse if the tangent C_iC_{i+1} misses the line segment AB ; it is a hyperbola if the tangent intersects AB between A and B. (To avoid the line passing through A or B we should insist that none of the A_i lie on the circle containing the B_i, and vice versa.) The second claim thereby reduces to a theorem that seems as if it should have been known a century ago, for which it seems to be easier to find a proof than a reference. The editor J. Chris Fisher now poses this proof as problem 3945, which appears in this issue of *Crux*.

3846. *Proposed by Arkady Alt.*

Let r be a positive real number. Prove that the inequality

$$\frac{1}{1+a+a^2}+\frac{1}{1+b+b^2}+\frac{1}{1+c+c^2} \geq \frac{3}{1+r+r^2}$$

holds for any positive a, b, c such that $abc = r^3$ if and only if $r \geq 1$.

We present the proof by the proposer, modified and expanded by the editor.

We first prove the following lemma :

Lemma. Let r be a given positive number. Then

$$\frac{1}{1+a+a^2}+\frac{1}{1+b+b^2} \geq \frac{2}{1+r+r^2} \tag{1}$$

for any positive a and b with $ab = r^2$ if and only if $r \geq r_0$, where r_0 is the unique positive root of the equation $4x^3+3x^2-3x-1=0$.

[Editor : Let $f(x) = 4x^3+3x^2-3x-1$. Then $f(0) = -1 < 0$ and $f(1) = 3 > 0$, so f has a real root $r_0 \in (0,1)$. By Rule of signs, r_0 is the only positive root.]

Proof. Note that if (1) holds for any positive a and b with $ab = r^2$, then

$$\lim_{a\to\infty}\left(\frac{1}{1+a+a^2}+\frac{1}{1+b+b^2}\right) = 1 \geq \frac{2}{1+r+r^2}$$

if and only if $r^2+r-1 \geq 0$, so $r \geq \dfrac{\sqrt{5}-1}{2}$.

Now, suppose $a, b > 0$ with $ab = r^2$, where $r \geq \dfrac{\sqrt{5}-1}{2}$. Let $x = a + b$, then $x \geq 2\sqrt{ab} = 2r$, and

$$\begin{aligned}
&\frac{1}{1+a+a^2} + \frac{1}{1+b+b^2} - \frac{2}{1+r+r^2} \\
&\qquad = \frac{2+a+b+a^2+b^2}{1+a+b+ab+ab(a+b)+a^2+b^2+a^2b^2} - \frac{2}{1+r+r^2} \\
&\qquad = \frac{2+x+x^2-2r^2}{1+x+r^2+(r^2)x+x^2-2(r^2)+r^4} - \frac{2}{1+r+r^2} \\
&\qquad = \frac{x^2+x+2-2r^2}{x^2+r^2x+x-r^2+r^4+1} - \frac{2}{1+r+r^2} \\
&\qquad = \frac{P(x)}{Q(x)},
\end{aligned}$$

where $Q(x) = (x^2+r^2x+x-r^2+r^4+1)(1+r+r^2)$ and by tedious computations together with synthetic division, we have :

$$\begin{aligned}
P(x) &= (x^2+x+2-2r^2)(1+r+r^2) - 2(x^2+r^2x+x-r^2+r^4+1) \\
&= (r^2+r-1)x^2 + (-r^2+r-1)x + 2(1-r^2)(r^2+r+1) - 2(1-r^2+r^4) \\
&= (r^2+r-1)x^2 - (r^2-r+1)x - 4r^4 - 2r^3 + 2r^2 + r \\
&= (x-2r)((r^2+r-1)x + 2r^3 + r^2 - r - 1).
\end{aligned}$$

Since $x \geq 2r$ and clearly $Q(x) > 0$, we have $\dfrac{P(x)}{Q(x)} \geq 0$ if and only if

$$(r^2+r-1)x + 2r^3 + r^2 - r - 1 \geq 0. \tag{2}$$

Since $x \geq 2r$ and $r^2+r-1 \geq 0$, (2) holds if and only if it holds for $x = 2r$; that is, $2r(r^2+r-1)+2r^3+r^2-r-1 \geq 0$ or $4r^3+3r^2-3r-1 \geq 0$ or $4r^2+3r-3-\frac{1}{r} \geq 0$.

The function $g(r) = 4r^2+3r-3-\frac{1}{r}$ is increasing on $(0,\infty)$ and $g(\frac{1}{2}) = -\frac{5}{2} < 0$, $g(\frac{3}{4}) = \frac{1}{6} > 0$, so it has only one root r_0 and $r_0 \in (\frac{1}{2}, \frac{3}{4})$. Hence, r_0 is the smallest value of r such that (2) holds for all $x \geq 2r$.

Furthermore, if we set $r_1 = \frac{\sqrt{5}-1}{2}$, then

$$4r_1^3+3r_1^2-3r_1-1 = 4r_1(r_1^2+r_1-1)-r^2+r_1-1+2(r_1-1) = 2(r_1-1) = \sqrt{5}-3 < 0,$$

so $r_1 < r_0 < \dfrac{3}{4}$.

In particular, (1) holds for all $a, b > 0$ such that $ab = r^2$ if $r \geq 1$ and this completes the proof of the lemma. ■

Using this lemma, we now prove that for all $a, b, c > 0$ with $abc = r^3$,

$$\frac{1}{1+a+a^2} + \frac{1}{1+b+b^2} + \frac{1}{1+c+c^2} \geq \frac{3}{1+r+r^2}$$

if and only if $r \geq 1$.

Necessity. Setting $c = \frac{r^3}{x^2}$ and $a = b = n$ in the given inequality where n is an arbitrary natural number, and passing to the limit, we have

$$1 = \lim_{n\to\infty}\left(\frac{2}{1+n+n^2} + \frac{1}{1+\frac{r^3}{n^2}+\frac{r^6}{n^4}}\right) \geq \frac{3}{1+r+r^2},$$

which implies $r^2 + r - 2 \geq 0$ or $(r+2)(r-1) \geq 0$, so $r \geq 1$.

Sufficiency. Let $a, b, c > 0$ with $abc = r^3, r \geq 1$. Without loss of generality, assume that $a \geq b \geq c$. Then $c^3 \geq abc = r^3$, so $c \geq r$. Set $x = \sqrt{ab}$. Then $c = \frac{r^3}{x^2}$ and $x \geq c$, so $x \geq r \geq 1$. Since $ab = x^2$, we have, by the lemma, that

$$\frac{1}{1+a+a^2} + \frac{1}{1+b+b^2} \geq \frac{2}{1+x+x^2}.$$

Hence, it suffices to prove that for any x and r with $x \geq r \geq 1$, we have

$$\frac{2}{1+x+x^2} + \frac{1}{1+\frac{r^3}{x^2}+\frac{r^6}{x^4}} \geq \frac{3}{1+r+r^2}. \tag{3}$$

Let

$$D(x) = \frac{2}{1+x+x^2} + \frac{1}{1+\frac{r^3}{x^2}+\frac{r^6}{x^4}} - \frac{3}{1+r+r^2}.$$

Then

$$D(x) = \frac{1}{1+\frac{r^3}{x^2}+\frac{r^6}{x^4}} - \frac{1}{1+r+r^2} - 2\left(\frac{1}{1+r+r^2} - \frac{1}{1+x+x^2}\right) \tag{4}$$

$$= \frac{r+r^2-\frac{r^3}{x^2}-\frac{r^6}{x^4}}{(1+\frac{r^3}{x^2}+\frac{r^6}{x^4})(1+r+r^2)} - \frac{x+x^2-r-r^2}{(1+r+r^2)(1+x+x^2)} = \frac{A(x)}{1+r+r^2}, \tag{5}$$

where

$$A(x) = \frac{x^2r\left(x^2-r^2\right)+r^2\left(x^4-r^4\right)}{x^4+x^2r^3+r^6} - \frac{2\left(1+x+r\right)\left(x-r\right)}{1+x+x^2} \tag{6}$$

$$= \frac{(x^2-r^2)(rx^2+r^2(x^2+r^2))(x^2+x+1) - 2(x+r+1)(x-r)(x^4+r^3x^2+r^6)}{(x^4+x^2r^3+r^6)(x^2+x+1)} \tag{7}$$

$$= \frac{(x-r)B(x)}{(x^4+x^2r^3+r^6)(x^2+x+1)}, \tag{8}$$

where

$$\begin{aligned}
B(x) &= (x+r)(x^2+x+1)(rx^2+r^2(x^2+r^2)) - 2(x+r+1)(x^4+r^3x^2+r^6)\\
&= (rx+r^2)(x^2+r(x^2+r^2))(x^2+x+1) - 2(x+r+1)(x^4+r^3x^2+r^6)\\
&= (r^2+r-2)x^5 + (r^3+2r^2-r-2)x^4 + (r^4-r^3+2r^2+r)x^3\\
&\quad + (r^5-r^4-r^3+r^2)x^2 + (-2r^6+r^5+r^4)x - 2r^7 - 2r^6 + r^5\\
&= (x-r)E(x),
\end{aligned}$$

where

$$E(x) = (r^2 + r - 2)x^4 + (2r^3 + 3r^2 - 3r - 2)x^3 + (3r^4 + 2r^3 - r^2 - r)x^2 + (4r^5 + r^4 - 2r^3)x + (2r^6 + 2r^5 - r^4).$$

It then suffices to prove that $E(x) \geq 0$ for all $x \geq r$. Since

$$\begin{aligned} E(r) &= r^6 + r^5 - 2r^4 + 2r^6 + 3r^5 - 3r^4 - 2r^3 + 3r^6 + 2r^5 - r^4 - r^3 \\ &\quad + 4r^6 + r^5 - 2r^4 + 2r^6 - 2r^5 - r^4 \\ &= r^7 + 12r^6 + 8r^5 - 9r^4 - 4r^3 \\ &= r^7 + 3r^6 + 4r^5 + r^3(r-1)(9r+4)(r+1) \geq 0 \end{aligned}$$

for $r \geq 1$ and since all the coefficients of $E(x)$ are clearly nonnegative as well, we conclude that $E(x) \geq E(r) > 0$ for all $x \geq r$. Hence, $B(x) \geq 0$ and from (8) $A(x) \geq 0$ and finally from (5) $D(x) \geq 0$, which establishes (3) and completes the proof.

Editor's comment. Perfetti's solution was computer assisted and Pranesachar's solution used Maple.

3847. *Proposed by Jung In Lee.*

Prove that there are no distinct positive integers a, b, c and nonnegative integer k that satisfy the conditions

$$a^{b+k} \mid b^{a+k}, \quad b^{c+k} \mid c^{b+k}, \quad c^{a+k} \mid a^{c+k}.$$

We present the solution by Joseph DiMuro.

We prove the stronger result that there are no distinct positive integers a, b, c and nonnegative real number k that satisfy the conditions

$$a^{b+k} \leq b^{a+k}, \;\; b^{c+k} \leq c^{b+k}, \;\; c^{a+k} \leq a^{c+k}. \tag{1}$$

Suppose (1) holds. Then from $a^{b+k} \leq b^{a+k}$, we have

$$\ln(a^{b+k}) \leq \ln(b^{a+k}) \;\text{ or }\; (b+k)\ln a \leq (a+k)\ln b,$$

so

$$\frac{\ln a}{a+k} \leq \frac{\ln b}{b+k}.$$

Similarly, from the other inequalities in (1), we deduce that

$$\frac{\ln b}{b+k} \leq \frac{\ln c}{c+k} \;\text{ and }\; \frac{\ln c}{c+k} \leq \frac{\ln a}{a+k}.$$